

STRONG ORTHOGONALITY BETWEEN THE MÖBIUS FUNCTION AND NONLINEAR EXPONENTIAL FUNCTIONS IN SHORT INTERVALS

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ABSTRACT. Let $\mu(n)$ be the Möbius function, $e(z) = \exp(2\pi iz)$, x real and $2 \leq y \leq x$. This paper proves two sequences $(\mu(n))$ and $(e(n^k \alpha))$ are strongly orthogonal in short intervals. That is, if $k \geq 3$ being fixed and $y \geq x^{1-1/4+\varepsilon}$, then for any $A > 0$, we have

$$\sum_{x < n \leq x+y} \mu(n) e(n^k \alpha) \ll y(\log y)^{-A}$$

uniformly for $\alpha \in \mathbb{R}$.

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1. INTRODUCTION

Let $\mu(n)$ be the Möbius function, $e(z) = \exp(2\pi iz)$, $k \geq 1$ an integer, x real and $2 \leq y \leq x$. The following classical result proved by Davenport [2] for $k = 1$ and by Hua [3] for $k \geq 2$: for any $A > 0$, we have

$$\sum_{n \leq x} \mu(n) e(n^k \alpha) \ll x(\log x)^{-A} \tag{1.1}$$

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uniformly for $\alpha \in \mathbb{R}$. Using Heath-Brown's identity, the estimate of the exponential sum involving the Möbius function in short intervals

$$S_k(x, y; \alpha) = \sum_{x < n \leq x+y} \mu(n) e(n^k \alpha) \quad (1.2)$$

was first studied by Zhan [25]. For the case $k = 1$, Zhan [25] gave an upper bound of the form $y(\log y)^{-A}$ for any $A > 0$, which holds for $y \geq x^{2/3+\varepsilon}$, and then refined by Zhan [26] to $y \geq x^{5/8+\varepsilon}$. For the case $k = 2$, Liu and Zhan [13] first established a nontrivial estimate of $S_2(x, y; \alpha)$ for $y \geq x^{11/16+\varepsilon}$ and all $\alpha \in \mathbb{R}$. In [14], Lü and Lao improved this result to $y \geq x^{2/3+\varepsilon}$ which was as good as what was previously derived from the Generalized Riemann Hypothesis in [13]. For the case $k \geq 3$, the result for all α was first given by Liu and Zhan [11], and recently Huang and Wang [5] gave an improvement by combining with the method of Kumchev.

Following [20], we say two sequences (a_n) and (b_n) of complex numbers are *asymptotically orthogonal* (in short, “orthogonal”) if

$$\sum_{n \leq N} a_n b_n = o \left(\left(\sum_{n \leq N} |a_n|^2 \right)^{1/2} \left(\sum_{n \leq N} |b_n|^2 \right)^{1/2} \right) \quad (1.3)$$

as $N \rightarrow \infty$; and *strongly asymptotically orthogonal* (in short, “strongly orthogonal”) if

$$\sum_{n \leq N} a_n b_n = O_A \left((\log N)^{-A} \sum_{n \leq N} |a_n b_n| \right) \quad (1.4)$$

for every $A > 0$, uniformly for $N \geq 2$. The *Möbius randomness law* (see [6, §13.1]) asserts that the sequence $(\mu(n))$ should be orthogonal to any “reasonable” sequence. Sarnak has recently posed a more precise conjecture in this direction and we refer the reader to [19], [20] and [10] for recent developments on this theme. In particular, Sarnak [20, Conjecture 4] proposed to replace the condition “reasonable” by “bounded with zero topological entropy”. The bound (1.1) shows that the two sequences $(\mu(n))$ and $(e(n^k \alpha))$ are strongly orthogonal. Similarly, we can say two sequences (a_n) and (b_n) of complex numbers are *orthogonal in short intervals of exponent Δ* if

$$\sum_{x < n \leq x+y} a_n b_n = o \left(\left(\sum_{x < n \leq x+y} |a_n|^2 \right)^{1/2} \left(\sum_{x < n \leq x+y} |b_n|^2 \right)^{1/2} \right) \quad (1.5)$$

as $x \rightarrow \infty$, for $y \geq x^{1-\Delta+\varepsilon}$; and *strongly orthogonal in short intervals of exponent Δ* if

$$\sum_{x < n \leq x+y} a_n b_n = O_A \left((\log x)^{-A} \sum_{x < n \leq x+y} |a_n b_n| \right) \quad (1.6)$$

for every $A > 0$, uniformly for $2 \leq y \leq x$ and $y \geq x^{1-\Delta+\varepsilon}$.

In this paper, the question we seek to answer is how large the exponent Δ_k can be for two sequences $(\mu(n))$ and $(e(n^k \alpha))$ uniformly for all $\alpha \in [0, 1]$ in the general case $k \geq 3$.

That is, we deal with $S_k(x, y; \alpha)$ for all $\alpha \in [0, 1]$ and $k \geq 3$. We say that the exponent Δ_k is *admissible* if $(\mu(n))$ and $(e(n^k \alpha))$ are strongly orthogonal in short intervals of exponent Δ_k for all $\alpha \in [0, 1]$. So far, in [5] the authors show that one has the admissible exponent

$$\Delta_k = \begin{cases} \frac{1}{5}, & \text{if } k = 3; \\ \frac{1}{2k}, & \text{if } k \geq 4, \end{cases} \quad (1.7)$$

which will be very small with the increase of k . The main result of this paper shows that there are large admissible exponents Δ_k for all $k \geq 3$ being fixed.

Theorem 1.1. *Let $k \geq 3$. The exponent $\Delta_k = 1/4$ is admissible. That is, if $y = x^\theta$ with $3/4 < \theta \leq 1$, then for any $A > 0$, we have*

$$S_k(x, y; \alpha) \ll y(\log y)^{-A},$$

uniformly for $\alpha \in (-\infty, +\infty)$.

Remark 1. In contrast to the admissible exponents derived in the previous work cited above, this exponent is bounded away from zero as $k \rightarrow \infty$.

An estimate for

$$\sum_{x < n \leq x+y} \Lambda(n) e(n^k \alpha) \quad (1.8)$$

can be established by the same methods in this paper, where $\Lambda(n)$ is the von Mongoldt function. And then combined with the Hardy–Littlewood circle method, this enables us to give some short interval variants of Hua’s theorems in additive number theory [3], such as [11, Theorems 2 and 3] and [4, Theorems 2 and 3].

Notation. Throughout the paper, the letter ε denotes a sufficiently small positive real number, while c without subscript stands for an absolute positive constant; both of them may be different at each occurrence. For example, we may write

$$(\log x)^c (\log x)^c \ll (\log x)^c, \quad x^\varepsilon \ll y^\varepsilon.$$

Any statement in which ε occurs holds for each positive ε , and any implied constant in such a statement is allowed to depend on ε . The letter p , with or without subscripts, is reserved for prime numbers. In addition, as usual, $e(z)$ denotes $\exp(2\pi iz)$. We write $(a, b) = \gcd(a, b)$, and we use $m \sim M$ as an abbreviation for the condition $M < m \leq 2M$.

2. OUTLINE OF OUR METHOD

Let

$$P = L^{c_1}, \quad Q = x^{k-2} y^2 / P, \quad R = x^{k-1} y, \quad (2.1)$$

where here and in the sequel L stands for $\log x$, and the letter c with or without subscripts denotes positive constants which depend at most on k and A , fixed in advance. Write α in the form

$$\alpha = \frac{a}{q} + \lambda, \quad (a, q) = 1. \quad (2.2)$$

To estimate $S_k(x, y, \alpha)$ for α in $[0, 1]$, we divide $[0, 1]$ into three subsets according to the idea due to Pan [16]. Let P , Q and R be defined as in (2.1). It follows from Dirichlet’s

lemma on rational approximations that every $\alpha \in [0, 1]$ can be written as (2.2), with q, λ satisfying one of the following three conditions:

- (a) $q \leq P, |\lambda| \leq \frac{1}{R};$
- (b) $q \leq P, \frac{1}{R} < |\lambda| \leq \frac{1}{qQ};$
- (c) $P < q \leq Q, |\lambda| \leq \frac{1}{qQ}.$

Denote by \mathcal{A} , \mathcal{B} and \mathcal{C} the three subsets of α satisfying (a), (b) and (c) respectively. Then $[0, 1]$ is the disjoint union of \mathcal{A} , \mathcal{B} and \mathcal{C} .

In §4 and §5 we employ analytic methods to deal with the case $\alpha \in \mathcal{A} \cup \mathcal{B}$, and obtain the following.

Proposition 2.1. *Let $k \geq 3$ and $y = x^\theta$ with $7/12 < \theta \leq 1$. Then for any c_1 , we have*

$$S_k(x, y; \alpha) \ll yL^{-A}$$

holds uniformly for $\alpha \in \mathcal{A}$.

Proposition 2.2. *Let $k \geq 3$, and $y = x^\theta$ with $2/3 < \theta < 1$. Then for any c_1 , we have*

$$S_k(x, y; \alpha) \ll yL^{-A}$$

holds uniformly for $\alpha \in \mathcal{B}$.

To prove the propositions, we appeal to zero-density estimate of L -functions in short intervals, see Zhan [25]. Since there is no explicit formula for $\sum_{n \leq u} \mu(n)\chi(n)$ in terms of zeros of the L -function with χ being a primitive character modulo l and $l \leq P$, we may use the so-called *Hooley–Huxley contour* method as in [18]. For $\alpha \in \mathcal{A}$, we can use the partial summation formula to handle the exponential function $e(n^k \lambda)$. But for $\alpha \in \mathcal{B}$, we must employ the exponential integral to deal with the exponential function.

In §6 we follow Kumchev’s approach to handle the case $\alpha \in \mathcal{C}$, and get the following.

Proposition 2.3. *Let $k \geq 3$ and $y = x^\theta$ with $3/4 < \theta \leq 1$. Then there exists $c_1 > 0$ such that the estimate*

$$S_k(x, y; \alpha) \ll yL^{-A}$$

holds uniformly for $\alpha \in \mathcal{C}$.

In proving the above proposition, we make use of the results in Daemen [1] (see Lemma 3.8) and apply Kumchev’s method to estimate the exponential sums of type I and type II:

$$\sum_{m \sim M} a(m) \sum_{x < mn \leq x+y} e((mn)^k \alpha), \quad (2.3)$$

$$\sum_{m \sim M} a(m) \sum_{x < mn \leq x+y} b(n) e((mn)^k \alpha) \quad (2.4)$$

respectively, then appeal to Vaughan's identity, where $a(m) \ll \tau^c(m)$, $b(n) \ll \tau^c(n)$, and $\tau(n)$ is the divisor function.

It is easily seen that Theorem 1.1 follows from Propositions 2.1, 2.2 and 2.3.

3. PRELIMINARIES

Lemma 3.1. *Let $k \geq 3$ and $\eta_k = 1 - \frac{1}{k}$. Write $\alpha = \frac{a}{q} + \lambda$ with $(a, q) = 1$. Then for any $\varepsilon > 0$, we have*

$$S_k(x, y; \alpha) \ll q^{\eta_k + \varepsilon} \sum_{d|q} \max_{\chi_{q/d}} \left| \sum_{\substack{x/d < m \leq (x+y)/d \\ (m, q) = 1}} \mu(m) \chi(m) e(m^k d^k \lambda) \right|.$$

Here the implied constant is absolute.

Proof. It is analogous to the proof of [12, Lemma 2]. □

Lemma 3.2. *Let $N(\alpha, T, \chi)$ be the number of zeros of $L(s, \chi)$ in*

$$\operatorname{Re}(s) \geq \alpha \text{ and } |\operatorname{Im}(s)| \leq T,$$

where $\chi = \chi_l$ is a primitive character modulo l . For $\frac{1}{2} \leq \alpha \leq 1$, $T \geq 2$ and $l \geq 1$, we have

$$N(\alpha, T, \chi) \ll (lT)^{167(1-\alpha)^{3/2}} (\log lT)^{17}.$$

Proof. See [15, Corollary 12.5]. □

Lemma 3.3. *Use the notation in Lemma 3.2. Let*

$$N(\alpha, T, H, \chi) = N(\alpha, T + H, \chi) - N(\alpha, T, \chi).$$

Then for $\frac{1}{2} \leq \alpha \leq 1$, $T^{\frac{35}{108} + \varepsilon} \leq H \leq T$ and $l \geq 1$, we have

$$N(\alpha, T, H, \chi) \ll (lH)^{\frac{8}{3}(1-\alpha)} (\log lH)^{216}.$$

Proof. See [27, Theorem 3]. □

Lemma 3.4. *For $l \geq 1$, $L(s, \chi)$ has no zeros in*

$$\sigma \geq 1 - \frac{c_0}{\log l + (\log(T+2))^{\frac{4}{5}}} \quad \text{and} \quad |t| \leq T,$$

where $c_0 > 0$ is a constant, except the only exceptional zero $\tilde{\beta}$. And for $l \leq (\log T)^c$ no such exceptional zero exists.

Proof. See [17, Satz 6.2]. □

Lemma 3.5 (Vaughan's identity). *Let $U, V \geq 1$. Then for any $n > \max\{U, V\}$, we have*

$$\mu(n) = - \sum_{\substack{lmd=n \\ 1 \leq d \leq V \\ 1 \leq m \leq U}} \mu(d) \mu(m) + \sum_{\substack{lmd=n \\ d > V \\ m > U}} \mu(d) \mu(m). \quad (3.1)$$

Proof. See [6, Proposition 13.5]. □

Lemma 3.6. *Suppose that $1 \leq N < N' < 2x$, $N' - N > x^\varepsilon d$ and $(c, d) = 1$. Then for $j, \nu \geq 1$, we have*

$$\sum_{\substack{N \leq n \leq N' \\ n \equiv c \pmod{d}}} \tau_j(n)^\nu \ll \frac{N' - N}{\varphi(d)} (\log N)^{j^\nu - 1},$$

the implied constant depending on ε, j , and ν at most.

Proof. See [21, Theorem 1]. □

When $k \geq 3$, we define the multiplicative function $w_k(q)$ by

$$w_k(p^{ku+v}) = \begin{cases} kp^{-u-1/2}, & \text{if } u \geq 0, v = 1, \\ p^{-u-1}, & \text{if } u \geq 0, v = 2, \dots, k. \end{cases}$$

By the argument of [23, Theorem 4.2], we have

$$S(q, a) = \sum_{1 \leq x \leq q} e(ax^k/q) \ll qw_k(q) \ll q^{1-1/k} \quad (3.2)$$

whenever $k \geq 3$ and $(a, q) = 1$. We also need several estimates for sums involving the function $w_k(q)$. We list those in the following lemma.

Lemma 3.7. *Let $w_k(q)$ be the multiplicative function defined above. Then the following inequalities hold for any fixed $\varepsilon > 0$:*

$$\sum_{n \sim N} \tau^c(n) w_k\left(\frac{q}{(q, n^j)}\right) \ll q^\varepsilon (\log N)^C w_k(q) N \quad (1 \leq j \leq k), \quad (3.3)$$

where $\tau(q)$ is the divisor function and C is a constant depending on c ;

$$\sum_{\substack{n \sim N \\ (n, h) = 1}} \tau^c(n) \tau^c(n+h) w_k\left(\frac{q}{(q, R(n, h))}\right) \ll q^\varepsilon (\log N)^C w_k(q) N + q^\varepsilon, \quad (3.4)$$

where $R(n, h) = ((n+h)^k - n^k)/h$.

Proof. See Lemma 2.3 and inequality (3.11) in Kawada and Wooley [8] and combine with the result in Lemma 3.6. □

Lemma 3.8. *Let $k \geq 3$ be an integer and $\gamma \geq 3$ be a real number. Let $0 < \rho \leq \sigma_k/\gamma$, where $\sigma_k = \frac{1}{2k(k-1)}$. Suppose that $y \leq x$, and $y \geq x^{\frac{\gamma}{2\gamma - \sigma_k - 1}}$. Then either*

$$\sum_{x < n \leq x+y} e(n^k \alpha) \ll y^{1-\rho+\varepsilon}, \quad (3.5)$$

or there exist integers a and q such that

$$1 \leq q \leq y^{k\rho}, \quad (a, q) = 1, \quad |q\alpha - a| \leq x^{1-k} y^{k\rho-1}, \quad (3.6)$$

and

$$\sum_{x < n \leq x+y} e(n^k \alpha) \ll y^{1-\rho+\varepsilon} + \frac{w_k(q)y}{1 + yx^{k-1}|\alpha - a/q|}. \quad (3.7)$$

Proof. Take

$$P_0 = y^{1/\gamma}, \quad Q_0 = x^{k-2}y^2/P_0.$$

By Dirichlet's lemma on rational approximations, there exists integers a and q with

$$1 \leq q \leq Q_0, \quad (a, q) = 1, \quad |q\alpha - a| \leq 1/Q_0. \quad (3.8)$$

When $q > P_0$, we rewrite the sum on the left of (3.5) as

$$\sum_{1 \leq n \leq z} e(\alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_0),$$

where $z \leq y$ and $\alpha_j = \binom{k}{j} \alpha u^{k-j}$, with u a fixed integer. Hence, it follows from the argument underlying the proof of [1, equation (3.5)] and [24, equation (4.23)] that

$$\sum_{x < n \leq x+y} e(n^k \alpha) \ll y P_0^{-\frac{1}{2k(k-1)} + \varepsilon} \ll y^{1-\rho+\varepsilon}. \quad (3.9)$$

When $q \leq P_0$, from (3.2) and [1, equations (5.1)-(5.5) and §6], we deduce

$$\sum_{1 \leq n \leq y} e(\alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_0) \ll \frac{w_k(q)y}{1 + yx^{k-1}|\alpha - a/q|} + \Delta,$$

where

$$\Delta \ll P_0^{1/2+\varepsilon} \left(1 + \frac{P_0 x^k}{x^{k-2}y^2}\right)^{1/2} \ll P_0^{1+\varepsilon} x/y \ll y^{1-\rho+\varepsilon},$$

provided that $y \geq x^{\frac{\gamma}{2\gamma-\sigma_k-1}}$. Thus, at least one of (3.5) and (3.7) holds. The lemma follows on noting that when conditions (3.6) fail, inequality (3.5) follows from (3.7). \square

4. THE CASE $\alpha \in \mathcal{A}$

Let

$$S_k(\chi) = \sum_{\substack{x_1 < m \leq x_1 + y_1 \\ (m, q) = 1}} \mu(m) \chi(m) e(m^k d^k \lambda), \quad (4.1)$$

where $\chi = \chi_l$ is a primitive character, $ld|q$, $x_1 = x/d$ and $y_1 = y/d$. By Lemma 3.1, in order to prove Propositions 2.1 and 2.2 it is sufficient to establish that

$$S_k(\chi) \ll q^{-1} y L^{-A}. \quad (4.2)$$

It follows from the main theorem in Ramachandra [18] that

$$\sum_{\substack{x_1 < m \leq u \\ (m, q) = 1}} \mu(m) \chi(m) \ll (u - x_1) \exp(-c(\log x_1)^{1/6}) \quad (4.3)$$

holds for $x_1^{7/12+\varepsilon} \leq u - x_1 \leq y_1$. Hence for $\alpha \in \mathcal{A}$, (4.2) can be proved by the partial summation formula. Recall that $|\lambda| \leq \frac{1}{R}$, we have

$$\begin{aligned}
S_k(\chi) &= \int_{x_1}^{x_1+y_1} e(\lambda d^k u^k) d \left(\sum_{\substack{x_1 < m \leq u \\ (m,q)=1}} \mu(m) \chi(m) \right) \\
&\ll y_1 \exp(-c(\log x_1)^{1/6}) + \int_{x_1}^{x_1+x_1^{7/12+\varepsilon}} (u - x_1) |\lambda| u^{k-1} du \\
&\quad + \int_{x_1+x_1^{7/12+\varepsilon}}^{x_1+y_1} (u - x_1) \exp(-c(\log x_1)^{1/6}) |\lambda| u^{k-1} du \\
&\ll x_1^{7/12+\varepsilon} + y \exp(-c' L^{1/6}) \ll q^{-1} y L^{-A}.
\end{aligned}$$

This proves Proposition 2.1.

5. THE CASE $\alpha \in \mathcal{B}$

Recall that for $\alpha \in \mathcal{B}$, we have $\frac{1}{R} < |\lambda| \leq \frac{1}{qQ}$. We start with Perron's summation formula (see [7, §5, Theorem 1]). For $x_1 < u < 2x_1$

$$\sum_{\substack{x_1 < m \leq u \\ (m,q)=1}} \mu(m) \chi(m) = \frac{1}{2\pi i} \int_{b_0-iT}^{b_0+iT} F(s, \chi) \frac{u^s - x_1^s}{s} ds + O\left(\frac{x_1 L}{T}\right), \quad (5.1)$$

where $b_0 = 1 + \frac{1}{L}$ and

$$F(s, \chi) = \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \frac{\mu(m) \chi(m)}{m^s}, \quad \operatorname{Re}(s) > 1. \quad (5.2)$$

Let $H(s, \chi) = \prod_{p|q} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$, then $F(s, \chi) = H(s, \chi)/L(s, \chi)$. Moreover, let $\rho(q)$ be defined by

$$\rho(q) = \prod_{p|q} \left(1 + \frac{1}{\sqrt{p}}\right). \quad (5.3)$$

Then we have

$$F(\sigma + it, \chi) \ll \frac{1}{|L(\sigma + it, \chi)|} \prod_{p|q} \left(1 - \frac{1}{\sqrt{p}}\right)^{-1} \ll \frac{L\rho(q)}{|L(\sigma + it, \chi)|}$$

for $\operatorname{Re}(s) = \sigma > \frac{1}{2}$.

Let M be the so-called *Hooley-Huxley contour* as described by Ramachandra [18]. Briefly speaking, we take the rectangle

$$\frac{1}{2} \leq \sigma \leq 1, \quad |t| \leq T + 2000(\log T)^2,$$

and divide it into equal rectangles of height $400(\log T)^2$ (the real line cuts one of these rectangles into two equal parts, we denote this rectangle by R_0). Let R_n ($n = -n_1, \dots, n_1$) be all these rectangles. In R_n , we fix a new right-hand side and obtain a new rectangle as follows. Consider R_{n-1}, R_n , and R_{n+1} whenever all of the three are defined. Pick out a zero of $L(s, \chi)$ in $R_{n-1} \cup R_n \cup R_{n+1}$ with the greatest real part β_n and $\operatorname{Re}(s) = \beta_n$ is the new right-hand side of R_n . Now we join all the right edges of the new rectangles by horizontal lines. These form the contour M' .

The Hooley–Huxley contour is obtained by making the following changes on M' . Let a , b , and ϑ be positive constant to be chosen later, satisfying $0 < \vartheta < 1$, a should be small and b should be close to 1. If $\beta_n < \vartheta$, then in place of β_n we take $\beta'_n = \beta_n + 3a(1 - \beta_n)$. If $\beta_n \geq \vartheta$, then β_n is replaced by $\beta'_n = \beta_n + b(1 - \beta_n)$. These form the Hooley–Huxley contour.

Now we join the points $b_0 \pm iT$ to M by horizontal lines H_1 and H_2 . The parameter T will be chosen as a suitable power of x . Since

$$\frac{1}{|L(s, \chi)|} \ll T^\varepsilon$$

for s on H_1 and H_2 as shown in [18], shifting the integral line in (5.1) to M we obtain

$$\sum_{\substack{x_1 < m \leq u \\ (m, q) = 1}} \mu(m) \chi(m) = \frac{1}{2\pi i} \int_M F(s, \chi) \frac{u^s - x_1^s}{s} ds + O\left(\frac{x_1 L}{T^{1-\varepsilon}}\right). \quad (5.4)$$

Therefore

$$\begin{aligned} S_k(\chi) &= \int_{x_1}^{x_1+y_1} e(\lambda d^k u^k) d \sum_{\substack{x_1 < m \leq u \\ (m, q) = 1}} \mu(m) \chi(m) \\ &= \frac{1}{2\pi i} \int_{x_1}^{x_1+y_1} e(\lambda d^k u^k) du \int_M F(s, \chi) u^{s-1} ds + O\left(\frac{1 + |\lambda| x^{k-1} y}{T^{1-\varepsilon}} x_1 L\right). \end{aligned} \quad (5.5)$$

Taking

$$T^{1-\varepsilon} = (1 + |\lambda| x^{k-1} y) q x y^{-1} L^{A+1}, \quad (5.6)$$

we have

$$S_k(\chi) = \frac{1}{2\pi i} \int_M F(s, \chi) ds \int_{x_1}^{x_1+y_1} u^{s-1} e(\lambda d^k u^k) du + O(q^{-1} y L^{-A}). \quad (5.7)$$

Let

$$\begin{aligned} I &:= \int_{x_1}^{x_1+y_1} u^{s-1} e(\lambda d^k u^k) du \\ &= \int_{x_1}^{x_1+y_1} u^{\sigma-1} e\left(\lambda d^k u^k + \frac{t}{2k\pi} \log u^k\right) du \\ &= \frac{1}{k} \int_{x_1^k}^{(x_1+y_1)^k} v^{\frac{\sigma}{k}-1} e\left(\lambda d^k v + \frac{t}{2k\pi} \log v\right) dv. \end{aligned}$$

Let \mathcal{V} denote the interval $[x_1^k, (x_1 + y_1)^k]$, and

$$f(v) = \lambda d^k v + \frac{t}{2k\pi} \log v, \quad v \in \mathcal{V}.$$

Then we have

$$\begin{aligned} f'(v) &= \lambda d^k + \frac{t}{2k\pi v} \gg \frac{\min_{v \in \mathcal{V}} |t + 2k\pi \lambda d^k v|}{x_1^k}, \\ f''(v) &= -\frac{t}{2k\pi v^2} \gg \frac{|t|}{x_1^{2k}}. \end{aligned}$$

Hence, we have (see Titchmarsh [22, Lemmas 4.3 and 4.4])

$$\begin{aligned} I &\ll x_1^{\sigma-k} \min \left(x_1^{k-1} y_1, \frac{x_1^k}{\min_{v \in \mathcal{V}} |t + 2k\pi \lambda d^k v|}, \frac{x_1^k}{\sqrt{|t|}} \right) \\ &= x_1^{\sigma-1} \min \left(y_1, \frac{x_1}{\min_{v \in \mathcal{V}} |t + 2k\pi \lambda d^k v|}, \frac{x_1}{\sqrt{|t|}} \right). \end{aligned}$$

Therefore

$$S_k(\chi) \ll \int_M \min \left(y_1, \frac{x_1}{\min_{v \in \mathcal{V}} |t + 2k\pi \lambda d^k v|}, \frac{x_1}{\sqrt{|t|}} \right) x_1^{\sigma-1} |F(s, \chi)| |ds| + q^{-1} y L^{-A}. \quad (5.8)$$

Take

$$H = \frac{x}{y} + 2^{k+2} k \pi |\lambda| x^{k-1} y. \quad (5.9)$$

For $|t + 2k\pi \lambda x^k| \leq H$, since $\frac{1}{R} < |\lambda| \leq \frac{1}{qQ}$, we have

$$\min \left(y_1, \frac{x_1}{\sqrt{|t|}} \right) \ll \min \left(y_1, \frac{x_1}{\sqrt{|\lambda| x^k}} \right)$$

holds. And the inequality

$$|t + 2k\pi \lambda x^k| \geq jH, \quad j \geq 1$$

ensures that, for $v \in \mathcal{V}$,

$$|t + 2k\pi \lambda d^k v| \geq jH - 2k\pi |\lambda| ((x+y)^k - x^k) \geq \frac{1}{2} jH. \quad (5.10)$$

Then we have

$$\begin{aligned}
& \int_M \min \left(y_1, \frac{x_1}{\min_{v \in \mathcal{V}} |t + 2k\pi\lambda d^k v|}, \frac{x_1}{\sqrt{|t|}} \right) x_1^{\sigma-1} |F(s, \chi)| |ds| \\
& \ll \int_{|t+2k\pi\lambda x^k| \leq H} \min \left(y_1, \frac{x_1}{\sqrt{|t|}} \right) x_1^{\sigma-1} |F(s, \chi)| |ds| \\
& \quad + \sum_{\substack{j \geq 1 \\ jH \leq 2T}} \int_{jH \leq |t+2k\pi\lambda x^k| \leq (j+1)H} \frac{x_1}{\min_{v \in \mathcal{V}} |t + 2k\pi\lambda d^k v|} x_1^{\sigma-1} |F(s, \chi)| |ds| \\
& \ll L \max_{|T_1| \leq 2T} \int_{T_1}^{T_1+H} \left(\min \left(y_1, \frac{x_1}{\sqrt{|\lambda|x^k}} \right) + \frac{x_1}{H} \right) x_1^{\sigma-1} |F(s, \chi)| |ds|.
\end{aligned}$$

For $\alpha \in \mathcal{B}$, by (5.9), it is a simple matter to show that

$$\min \left(y_1, \frac{x_1}{\sqrt{|\lambda|x^k}} \right) + \frac{x_1}{H} \ll y_1 + \frac{x_1}{H} \ll \sqrt{\frac{x_1 y_1}{H}} L^{c_1} = \frac{1}{d} \sqrt{\frac{xy}{H}} L^{c_1}.$$

Let $M(H)$ denote the part of M satisfying

$$T_1 \leq \text{Im}(s) \leq T_1 + H, \quad |T_1| \leq 2T.$$

Then

$$\begin{aligned}
S_k(\chi) & \ll L^{c_1+1} \sqrt{\frac{xy}{H}} \max_{|T_1| \leq 2T} \int_{M(H)} x^{\sigma-1} |F(s, \chi)| |ds| + q^{-1} y L^{-A} \\
& \ll L^{c_1+2} \rho(q) \sqrt{\frac{xy}{H}} \max_{|T_1| \leq 2T} \int_{M(H)} x^{\sigma-1} |L(s, \chi)|^{-1} |ds| + q^{-1} y L^{-A}.
\end{aligned}$$

Since $H \geq xy^{-1}$, we have $\sqrt{\frac{xy}{H}} \leq y$. To prove Proposition 2.2, now it is sufficient to show that for $|T_1| \leq 2T$,

$$\int_{M(H)} x^{\sigma-1} |L(s, \chi)|^{-1} |ds| \ll L^{-A-2c_1-2}. \quad (5.11)$$

To prove (5.11), we just follow the method of Ramachandra [18]. It is shown in [18, Lemma 5] that

$$\begin{aligned}
|L(s, \chi)|^{-1} & \ll T^\varepsilon, \text{ if } s \in M(H) \text{ and } \text{Re}(s) \leq \vartheta + b(1 - \vartheta), \\
|L(s, \chi)|^{-1} & \ll \exp((\log T)^{3(1-b)}), \text{ if } s \in M(H) \text{ and } \text{Re}(s) > \vartheta + b(1 - \vartheta).
\end{aligned}$$

We divide the smallest vertical strip containing $M(H)$ into vertical strips of width $1/\log T$. Consider the bits of $M(H)$, say $M(H, \sigma')$, in the vertical strip about the abscissa σ' . Then by the construction of the Hooley–Huxley contour, we have

$$\int_{M(H, \sigma')} |ds| \ll N(\sigma', T_1, H, \chi) (\log T)^{10},$$

where σ' is $\sigma + 3a(1 - \sigma)$ or $\sigma + b(1 - \sigma)$ according as $\sigma' \leq \vartheta$ or $\sigma' > \vartheta$. By the above discussion and Lemmas 3.2-3.4, we obtain

$$\begin{aligned}
& \int_{M(H)} x^{\sigma-1} |L(s, \chi)|^{-1} |ds| \\
&= \int_{\substack{M(H) \\ \sigma' < \vartheta}} x^{\sigma'-1} |L(s, \chi)|^{-1} |ds| + \int_{\substack{M(H) \\ \vartheta \leq \sigma' \leq \vartheta + b(1-\vartheta)}} x^{\sigma'-1} |L(s, \chi)|^{-1} |ds| \\
&+ \int_{\substack{M(H) \\ \sigma' > \vartheta + b(1-\vartheta)}} x^{\sigma'-1} |L(s, \chi)|^{-1} |ds| \quad (s = \sigma' + it) \\
&\ll T^\varepsilon \left(\frac{H^{\frac{8}{3}(1-3a)^{-1}}}{x} \right)^{(1-\vartheta)} + T^\varepsilon \left(\frac{T^{167(1-\vartheta)^{\frac{1}{2}}(1-b)^{-\frac{3}{2}}}}{x} \right)^{(1-b)(1-\vartheta)} \\
&+ \exp((\log T)^{3(1-b)}) \left(\frac{T^{167(1-\vartheta)^{\frac{1}{2}}(1-b)^{-\frac{3}{2}}}}{x} \right)^{c_0(\log T)^{-4/5}}
\end{aligned}$$

provided a , b , and ϑ satisfy

$$H^{\frac{8}{3}(1-3a)^{-1}} \leq x^{1-\varepsilon}, \quad (5.12)$$

$$T^{400(1-\vartheta)^{\frac{1}{2}}(1-b)^{-\frac{3}{2}}} \leq x. \quad (5.13)$$

In fact, we may first choose a such that

$$\frac{8}{3} \left(\frac{1}{3} + \varepsilon \right) \frac{1}{1-3a} < 1 - \varepsilon \quad (H \leq x^{\frac{1}{3}+\varepsilon}),$$

b such that $3(1-b) = \frac{1}{100}$ and then ϑ such that (5.13) holds. Hence

$$\int_{M(H)} x^{\sigma-1} |L(s, \chi)|^{-1} |ds| \ll \exp(-c'_0 L^{\frac{1}{6}}) \quad (c'_0 > 0)$$

and (5.11) follows. Thus, we prove Proposition 2.2.

6. THE CASE $\alpha \in \mathcal{C}$

6.1. Type I estimate. Recall that

$$y = x^\theta, \quad L = \log x.$$

The following lemma treats the exponential sums of type I which is an improvement of [5, Lemma 8].

Lemma 6.1. *Let $k \geq 3$ be an integer and $\gamma \geq 3$ be a real number. Let $0 < \rho < \sigma_k/(2\gamma)$, with $\sigma_k = \frac{1}{2k(k-1)}$. Suppose that $\alpha \in \mathcal{C}$ and $a(m) \ll \tau^c(m)$. Define*

$$\mathcal{T}_1 = \sum_{m \sim M} a(m) \sum_{x < mn \leq x+y} e((mn)^k \alpha).$$

Then for any $A > 0$, we have

$$\mathcal{T}_1 \ll y L^{-A},$$

provided that

$$M \ll y \left(\frac{y}{x} \right)^{\frac{\gamma}{\gamma - \sigma_k - 1}}, \quad M \ll yx^{-\gamma\rho/\sigma_k}, \quad M^{2k} \ll yx^{k-1-2k\rho}, \quad (6.1)$$

and

$$c_1 > (k+1)(A+C), \quad (6.2)$$

where C is a constant depending on c .

Proof. Set

$$S_m = \sum_{X < n \leq X+Y} e(m^k n^k \alpha),$$

where $X = x/m, Y = y/m$ with $m \sim M$. Define ν by $Y^\nu = x^\rho L^{-1}$. Note that, by (6.1), we have

$$\nu < \sigma_k/\gamma.$$

We denote by \mathcal{M} the set of integers $m \sim M$, for which there exist integers b_1 and r_1 with

$$1 \leq r_1 \leq Y^{k\nu}, \quad (b_1, r_1) = 1, \quad |r_1 m^k \alpha - b_1| \leq X^{1-k} Y^{k\nu-1}. \quad (6.3)$$

By (6.1), we have $Y \gg X^{\gamma/(2\gamma - \sigma_k - 1)}$. We apply Lemma 3.8 to the summation over n and get

$$S_m \ll Y^{1-\nu+\varepsilon} + \frac{w_k(r_1)Y}{1 + YX^{k-1}|m^k \alpha - b_1/r_1|},$$

for $m \in \mathcal{M}$. Consequently,

$$\begin{aligned} \mathcal{T}_1 &\ll \sum_{m \sim M} a(m) Y^{1-\nu+\varepsilon} + \sum_{m \in \mathcal{M}} \frac{a(m) w_k(r_1) Y}{1 + YX^{k-1}|m^k \alpha - b_1/r_1|} \\ &\ll x^{\theta-\rho+\varepsilon} + T_1(\alpha), \end{aligned}$$

where

$$T_1(\alpha) = \sum_{m \in \mathcal{M}} \frac{a(m) w_k(r_1) Y}{1 + YX^{k-1}|m^k \alpha - b_1/r_1|}.$$

We apply Dirichlet's lemma on rational approximations to find integers b and r with

$$1 \leq r \leq x^{-k\rho} Y X^{k-1}, \quad (b, r) = 1, \quad |r\alpha - b| \leq x^{k\rho} Y^{-1} X^{1-k}. \quad (6.4)$$

By (6.1), (6.3) and (6.4), we have

$$\begin{aligned} |b_1 r - b m^k r_1| &= |r(b_1 - r_1 m^k \alpha) + r_1 m^k (r\alpha - b)| \\ &\leq x^{-k\rho} Y X^{k-1} X^{1-k} Y^{k\nu-1} + Y^{k\nu} (2M)^k x^{k\rho} Y^{-1} X^{1-k} \\ &\ll L^{-k} + M^{2k} L^{-k} x^{2k\rho-k+1} y^{-1} \ll L^{-k} < 1, \end{aligned}$$

whence

$$\frac{b_1}{r_1} = \frac{m^k b}{r}, \quad r_1 = \frac{r}{(r, m^k)}.$$

Thus, by Lemma 3.7, we have

$$\begin{aligned} T_1(\alpha) &\ll \frac{yM^{-1}}{1 + yx^{k-1}|\alpha - b/r|} \sum_{m \sim M} \tau^c(m) w_k \left(\frac{r}{(r, m^k)} \right) \\ &\ll \frac{w_k(r) r^\varepsilon L^C y}{1 + yx^{k-1}|\alpha - b/r|}. \end{aligned}$$

Recall that b and r satisfy the conditions (6.4). We now consider three cases depending on the sizes of r and $|r\alpha - b|$.

Case 1: If $r > L^{(k+1)(A+C)}$, then $T_1(\alpha) \ll yL^{-A}$.

Case 2: If $r \leq L^{(k+1)(A+C)}$ and $|r\alpha - b| > y^{-1}x^{1-k}L^{(k+1)(A+C)}$, then $T_1(\alpha) \ll yL^{-A}$.

Case 3: If $r \leq L^{(k+1)(A+C)}$ and $|r\alpha - b| \leq y^{-1}x^{1-k}L^{(k+1)(A+C)}$, we have

$$\begin{aligned} |ra - bq| &= |r(a - q\alpha) + q(r\alpha - b)| \\ &\leq \frac{1}{Q}L^{(k+1)(A+C)} + Qy^{-1}x^{1-k}L^{(k+1)(A+C)} \\ &\leq \frac{PL^{(k+1)(A+C)}}{x^{k-2}y^2} + \frac{yL^{(k+1)(A+C)}}{xP}. \end{aligned}$$

By (6.2), we have $|ra - bq| < 1$. Hence

$$a = b, \quad q = r.$$

Then

$$T_1(\alpha) \ll \frac{w_k(q)q^\varepsilon L^C y}{1 + yx^{k-1}|\alpha - a/q|}.$$

So we have

$$\mathcal{T}_1 \ll yL^{-A} + \frac{w_k(q)q^\varepsilon L^C y}{1 + yx^{k-1}|\alpha - a/q|}.$$

For $\alpha \in \mathcal{C}$, we have $q > P = L^{c_1}$. If we have c_1 as in (6.2) then $\mathcal{T}_1 \ll yL^{-A}$. \square

Remark 2. One can estimate the following exponential sums of type I/II

$$\sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} a(m_1, m_2) \sum_{x < m_1 m_2 n \leq x+y} e((m_1 m_2 n)^k \alpha)$$

with some suitable conditions on M_1 and M_2 as [9, Lemma 3.2] and [24, Lemma 4.2] did, and give a better result than Lemma 6.1.

6.2. Type II estimate. To prove Theorem 1.1, we also need to handle the exponential sums of type II. Let $a(m)$ and $b(n)$ be arithmetic functions satisfying the property that for all natural numbers m and n , one has

$$a(m) \ll \tau^c(m) \quad \text{and} \quad b(n) \ll \tau^c(n). \quad (6.5)$$

Let M and N be positive parameters, and define the exponential sum $\mathcal{T}_2 = \mathcal{T}_2(\alpha; M)$ by

$$\mathcal{T}_2(\alpha; M) := \sum_{m \sim M} a(m) \sum_{x < mn \leq x+y} b(n) e((mn)^k \alpha). \quad (6.6)$$

The following lemma gives an estimate for \mathcal{T}_2 which is an improvement of [9, Lemma 3.1].

Lemma 6.2. *Let k, γ, σ_k be as in Lemma 6.1. Let $0 < \rho < \sigma_k/(8\gamma)$. Suppose that $\alpha \in \mathcal{C}$. And let x and y be positive numbers with*

$$y = x^\theta, \quad \frac{1}{(1-2\rho)} \frac{3\gamma - \sigma_k - 1}{2(2\gamma - \sigma_k - 1)} \leq \theta \leq 1. \quad (6.7)$$

Then

$$\mathcal{T}_2 \ll yL^{-A},$$

provided that

$$x^{1/2} \leq M \ll x^{\theta-2\rho}, \quad (6.8)$$

and

$$c_1 > 2(k+1)(A+C), \quad (6.9)$$

where C is a constant depending on c .

Proof. Set $N = x/M$, $X = x/N$, and $Y = y/N = yM/x$. Define ν by $Y^\nu = x^{2\rho}L^{-1}$. By (6.8), we have

$$\nu < \sigma_k/\gamma.$$

For $n_1, n_2 \leq 2N$, let

$$\mathcal{M}(n_1, n_2) = \{m \in (M, 2M] : x < mn_1, mn_2 \leq x + y\}.$$

By Cauchy's inequality and an interchange of the order of summation, we have

$$\mathcal{T}_2^2 \ll y^{1+\varepsilon}M + ML^C T_1(\alpha), \quad (6.10)$$

where

$$T_1(\alpha) = \sum_{n_1 < n_2} \tau^c(n_1) \tau^c(n_2) \left| \sum_{m \in \mathcal{M}(n_1, n_2)} e(\alpha(n_2^k - n_1^k)m^k) \right|.$$

Let \mathcal{N} denote the set of pairs (n_1, n_2) with $n_1 < n_2$ and $\mathcal{M}(n_1, n_2) \neq \emptyset$ for which there exist integers b and r such that

$$1 \leq r \leq Y^{k\nu}, \quad (b, r) = 1, \quad |r(n_2^k - n_1^k)\alpha - b| \leq Y^{k\nu-1}X^{1-k}. \quad (6.11)$$

Since $N/2 < n_1 < n_2 \leq 2N$ and $\mathcal{M}(n_1, n_2) \neq \emptyset$, we have $n_2 - n_1 \leq yx^{-1}n_1$. Hence $\#\mathcal{N} \ll xyM^{-2}$. In order to handle the inner summation in $T_1(\alpha)$, we set

$$X_1 := \max \left\{ M, \frac{x}{n_1} \right\} \asymp M = \frac{x}{N} = X,$$

$$Y_1 := \min \left\{ 2M, \frac{x+y}{n_2} \right\} - \max \left\{ M, \frac{x}{n_1} \right\} \ll \frac{y}{N} = Y.$$

If $Y_1 < X_1^{\gamma/(2\gamma-\sigma_k-1)}$, by (6.7) and (6.8), the contribution to $T_1(\alpha)$ is

$$\ll xyM^{-2}M^{\gamma/(2\gamma-\sigma_k-1)} \ll y^{2-2\rho+\varepsilon}M^{-1}.$$

If $Y_1 \geq X_1^{\gamma/(2\gamma-\sigma_k-1)}$, since $\nu < \sigma_k/\gamma$, we can apply Lemma 3.8 with $\rho = \nu$, $x = X_1$, and $y = Y_1$ to the inner summation in $T_1(\alpha)$. We obtain

$$T_1(\alpha) \ll y^{2-2\rho+\varepsilon} M^{-1} + T_2(\alpha), \quad (6.12)$$

where

$$\begin{aligned} T_2(\alpha) &= \sum_{(n_1, n_2) \in \mathcal{N}} \frac{\tau^c(n_1) \tau^c(n_2) w_k(r) Y_1}{1 + Y_1 X_1^{k-1} |(n_2^k - n_1^k) \alpha - b/r|} \\ &\ll \sum_{(n_1, n_2) \in \mathcal{N}} \frac{\tau^c(n_1) \tau^c(n_2) w_k(r) Y}{1 + Y X^{k-1} |(n_2^k - n_1^k) \alpha - b/r|}. \end{aligned}$$

We now change the summation variables in $T_2(\alpha)$ to

$$d = (n_1, n_2), \quad n = n_1/d, \quad h = (n_2 - n_1)/d.$$

We obtain

$$T_2(\alpha) \ll \sum_{dh \leq y/M} \sum_n' \frac{\tau^c(nd) \tau^c(nd + hd) w_k(r) Y}{1 + Y X^{k-1} |hd^k R(n, h) \alpha - b/r|}, \quad (6.13)$$

where $R(n, h) = ((n+h)^k - n^k)/h$ and the inner summation is over n with $(n, h) = 1$ and $(nd, (n+h)d) \in \mathcal{N}$. For each pair (d, h) appearing in the summation on the right-hand side of (6.13), Dirichlet's lemma on rational approximations yields integers b_1 and r_1 with

$$1 \leq r_1 \leq x^{-2k\rho} Y X^{k-1}, \quad (b_1, r_1) = 1, \quad |r_1 h d^k \alpha - b_1| \leq x^{2k\rho} Y^{-1} X^{1-k}. \quad (6.14)$$

As $R(n, h) \leq 4^k (N/d)^{k-1}$, combining (6.8), (6.11) and (6.14), we have

$$\begin{aligned} |b_1 r R(n, h) - b r_1| &= |r R(n, h) (b_1 - r_1 h d^k \alpha) + r_1 (r h d^k R(n, h) \alpha - b)| \\ &\leq r_1 Y^{k\nu-1} X^{1-k} + r R(n, h) x^{2k\rho} Y^{-1} X^{1-k} \\ &\leq L^{-k} + 4^k N^{k-1} x^{2k\rho} L^{-k} x^{2k\rho} Y^{-1} X^{1-k} < 1. \end{aligned}$$

Hence,

$$\frac{b}{r} = \frac{b_1 R(n, h)}{r_1}, \quad r = \frac{r_1}{(r_1, R(n, h))}. \quad (6.15)$$

Combining (6.13) and (6.15), we obtain

$$T_2(\alpha) \ll \sum_{dh \leq y/M} \frac{\tau^{2c}(d) Y}{1 + Y X^{k-1} N_d^{k-1} |h d^k \alpha - b_1/r_1|} \sum_{\substack{n \sim N_d \\ (n, h)=1}} \tau^c(n) \tau^c(n+h) w_k \left(\frac{r_1}{(r_1, R(n, h))} \right),$$

where $N_d = N/d$. By Lemma 3.7, we deduce that

$$T_2(\alpha) \ll y^2 x^{-1+\varepsilon} + T_3(\alpha), \quad (6.16)$$

where

$$\begin{aligned} T_3(\alpha) &= \sum_{dh \leq y/M} \frac{w_k(r_1) r_1^\varepsilon L^c \tau^c(d) Y N_d}{1 + Y X^{k-1} N_d^{k-1} |h d^k \alpha - b_1/r_1|} \\ &\ll \sum_{dh \leq y/M} \frac{r_1^\varepsilon L^c \tau^c(d) Y N_d}{(r_1 + Y X^{k-1} N_d^{k-1} |r_1 h d^k \alpha - b_1|)^{1/k}}. \end{aligned}$$

We now write \mathcal{H} for the set of pairs (d, h) with $dh \leq y/M$ for which there exist integers b_1 and r_1 subject to

$$1 \leq r_1 \leq x^{2k\rho}, \quad (b_1, r_1) = 1, \quad |r_1 h d^k \alpha - b_1| \leq x^{-k+1+2k\rho} Y^{-1}. \quad (6.17)$$

We have

$$T_3(\alpha) \ll y^{2-2\rho+\varepsilon} M^{-1} + T_4(\alpha), \quad (6.18)$$

where

$$T_4(\alpha) = \sum_{(d,h) \in \mathcal{H}} \frac{w_k(r_1) r_1^\varepsilon L^c \tau^c(d) Y N_d}{1 + Y X^{k-1} N_d^{k-1} |h d^k \alpha - b_1/r_1|}.$$

For each $d \leq y/M$, Dirichlet's lemma on rational approximations yields integers b_2 and r_2 with

$$1 \leq r_2 \leq x^{k-1-2k\rho} Y/2, \quad (b_2, r_2) = 1, \quad |r_2 d^k \alpha - b_2| \leq 2x^{-k+1+2k\rho} Y^{-1}. \quad (6.19)$$

Combining (6.17) and (6.19), we obtain

$$\begin{aligned} |b_2 r_1 h - b_1 r_2| &= |r_1 h (b_2 - r_2 d^k \alpha) + r_2 (r_1 h d^k \alpha - b_1)| \\ &\leq r_1 h |r_2 d^k \alpha - b_2| + r_2 |r_1 h d^k \alpha - b_1| \\ &\leq 1/2 + 2x^{-k+2+4k\rho} M^{-2} < 1, \end{aligned}$$

whence

$$\frac{b_1}{r_1} = \frac{h b_2}{r_2}, \quad r_1 = \frac{r_2}{(r_2, h)}.$$

We write $Z_d = Y X^{k-1} N_d^{k-1} |d^k \alpha - b_2/r_2|$ and by Lemma 3.7, we obtain

$$\begin{aligned} T_4(\alpha) &= \sum_{(d,h) \in \mathcal{H}} \frac{r_2^\varepsilon L^c \tau^c(d) Y N_d}{1 + Z_d h} w_k \left(\frac{r_2}{(r_2, h)} \right) \\ &\ll \sum_{d \leq y/M} r_2^\varepsilon L^c \tau^c(d) y d^{-1} L \max_{1 \leq H \leq \frac{y}{M d}} \sum_{h \sim H} \frac{1}{1 + Z_d h} w_k \left(\frac{r_2}{(r_2, h)} \right) \\ &\ll \sum_{d \leq y/M} \frac{r_2^\varepsilon L^c \tau^c(d) w_k(r_2) y^2 M^{-1}}{d^2 (1 + y(M d)^{-1} Z_d)}. \end{aligned}$$

Hence

$$T_4(\alpha) \ll y^{2-2\rho+\varepsilon} M^{-1} + T_5(\alpha), \quad (6.20)$$

where

$$T_5(\alpha) = \sum_{d \in \mathcal{D}} \frac{r_2^\varepsilon L^c \tau^c(d) w_k(r_2) y^2 M^{-1}}{d^2 (1 + y^2 x^{k-2} d^{-k} |d^k \alpha - b_2/r_2|)},$$

and \mathcal{D} is the set of integers $d \leq x^{2\rho}$ for which there exist integers b_2 and r_2 with

$$1 \leq r_2 \leq x^{2k\rho}, \quad (b_2, r_2) = 1, \quad |r_2 d^k \alpha - b_2| \leq y^{-2} x^{2-k} L^{(k+1)(2A+C)}. \quad (6.21)$$

Combining (2.1), (2.2) and (6.21), we deduce that

$$\begin{aligned} |r_2 d^k a - b_2 q| &= |r_2 d^k (a - q\alpha) + q(r_2 d^k \alpha - b_2)| \\ &\leq r_2 d^k Q^{-1} + q |r_2 d^k \alpha - b_2| \\ &\leq x^{4k\rho} Q^{-1} + y^{-2} x^{2-k} L^{(k+1)(2A+C)} Q < 1, \end{aligned}$$

whence

$$\frac{b_2}{r_2} = \frac{d^k a}{q}, \quad r_2 = \frac{q}{(q, d^k)}.$$

Thus, recalling Lemma 3.7, we obtain

$$\begin{aligned} T_5(\alpha) &\ll \frac{q^\varepsilon L^c y^2 M^{-1}}{1 + y^2 x^{k-2} |\alpha - a/q|} \sum_{d \leq x^{2\rho}} \tau^c(d) d^{-2} w_k \left(\frac{q}{(q, d^k)} \right) \\ &\ll \frac{q^\varepsilon L^C w_k(q) y^2 M^{-1}}{1 + y^2 x^{k-2} |\alpha - a/q|}. \end{aligned} \quad (6.22)$$

The desired estimate follows from (6.7), (6.8), (6.10), (6.12), (6.16), (6.18), (6.20), and (6.22). \square

6.3. Complete the proof of Proposition 2.3. We now deduce Proposition 2.3 from Lemmas 6.1, 6.2 and Vaughan's identity for $\mu(n)$.

We put

$$U = x^{\theta/2-\rho}, \quad V = x^{1-\theta+2\rho}. \quad (6.23)$$

Take

$$\rho = \frac{1}{2} \min \left\{ \frac{\sigma_k}{8\gamma}, \frac{1}{2} \left(\theta - \frac{2}{3} \right) \right\}. \quad (6.24)$$

We have

$$UV \asymp (x+y)/U \asymp x^{1-\theta/2+\rho} \ll x^{\theta-2\rho}. \quad (6.25)$$

And then we apply Vaughan's identity as in Lemma 3.5. Thus we deduce that

$$S_k(x, y; \alpha) = -S_1 + S_2, \quad (6.26)$$

where

$$\begin{aligned} S_1 &= \sum_{1 \leq v \leq UV} \lambda_0(v) \sum_{x < lv \leq x+y} e((lv)^k \alpha), \\ S_2 &= \sum_{V < u \leq (x+y)/U} \lambda_1(u) \sum_{\substack{x < mu \leq x+y \\ m > U}} \mu(m) e((mu)^k \alpha), \end{aligned}$$

in which

$$\lambda_0(v) = \sum_{\substack{md=v \\ 1 \leq d \leq V \\ 1 \leq m \leq U}} \mu(d)\mu(m) \quad \text{and} \quad \lambda_1(u) = \sum_{\substack{d|u \\ d > V}} \mu(d).$$

We begin with estimating the sum S_2 . Take

$$\gamma = (\theta - 3/4)^{-1}. \quad (6.27)$$

Since $3/4 < \theta \leq 1$, by (6.24) we have

$$\frac{1}{(1-2\rho)} \frac{3\gamma - \sigma_k - 1}{2(2\gamma - \sigma_k - 1)} \leq \theta \leq 1.$$

To apply Lemma 6.2, we further divide S_2 into two parts

$$S_{21} = \sum_{x^{1/2} \leq u \leq (x+y)/U} \lambda_1(u) \sum_{\substack{x < mu \leq x+y \\ m > U}} \mu(m) e((mu)^k \alpha),$$

and

$$S_{22} = \sum_{V < u < x^{1/2}} \lambda_1(u) \sum_{\substack{x < mu \leq x+y \\ m > U}} \mu(m) e((mu)^k \alpha).$$

On noting that (6.23), (6.25) and $\lambda_1(u) \leq \tau(u)$, we can divide the summation over u into dyadic intervals to deduce from Lemma 6.2 that

$$S_{21} \ll (\log x) \max_{x^{1/2} \leq M \leq (x+y)/U} \left| \sum_{u \sim M} a(u) \sum_{x < mu \leq x+y} b(m) e((mu)^k \alpha) \right| \ll yL^{-A},$$

where $a(u) = \lambda_1(u)$, and $b(m) = \mu(m)$ if $m > U$ and is 0 if else. For S_{22} , we first interchange the order of summation, and then by the same argument as above, we obtain

$$S_{22} \ll yL^{-A}.$$

Hence we obtain

$$S_2 \ll yL^{-A}. \quad (6.28)$$

Next we estimate S_1 . Write

$$S_3(Z, W) = \sum_{Z < v \leq W} \lambda_0(v) \sum_{x < lv \leq x+y} e((lv)^k \alpha).$$

Then we find that

$$S_1 = S_3(0, V) + S_3(V, UV). \quad (6.29)$$

Note that (6.23), (6.25) and the bound $|\lambda_0(v)| \leq \tau(v)$, we deduce from Lemma 6.2 that

$$S_3(V, UV) \ll yL^{-A}. \quad (6.30)$$

We then estimate $S_3(0, V)$. Since $3/4 < \theta \leq 1$, by (6.23), (6.24) and (6.27), we have

$$V \ll y \left(\frac{y}{x} \right)^{\frac{\gamma+1}{\gamma-\sigma_k-1}}, \quad V \ll yx^{-\gamma\rho/\sigma_k}, \quad V^{2k} \ll yx^{k-1-2k\rho}.$$

So we can divide the summation over v into dyadic intervals to deduce from Lemma 6.1 that

$$S_3(0, V) \ll yL^{-A}. \quad (6.31)$$

Thus, by combining (6.30) and (6.31), we deduce from (6.29) that

$$S_1 \ll yL^{-A}. \quad (6.32)$$

Proposition 2.3 follows from (6.26), (6.28) and (6.32).

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